Université d'Ottawa · University of Ottawa
Faculté des sciences
Mathématiques et de statistique
Faculty of Science
Mathematics and Statistics

Calculus III for Engineers
MAT 2322A - Fall 2015
Final Exam
Professor: Victor G. LeBlanc
Time limit: 3 hours. Closed books.

Name: Solutions

ID Number:

Instructions

- This is a closed book exam. Furthermore, all cell phones, pagers or any other electronic or communication devices are forbidden. **The only calculators which are allowed are Texas Instruments TI-30, TI-34, Casio fx-260 and fx-300, scientific and non programmable.**
- This exam has 15 pages and you have 3 hours to complete it.
- Read each question carefully before answering.
- Questions 1 to 10 are multiple choice questions. These questions are worth 2 points each and no partial marks are possible. **Please write your answers in the corresponding boxes in the grid below entitled "Answers to multiple choice Qs".**
- Questions 11 to 16 are long answer questions and are worth 5 marks each, so organize your time accordingly. **A correct answer requires a full, clearly-written and detailed solution.** Answer each question in the space provided, using backs of pages or the extra pages at the end if necessary.
- Do not unstaple the test. Good luck!

Answers to multiple choice Qs

1	2	3	4	5	6	7	8	9	10
B	A	C	F	C	B	D	C	F	E

Grid below is used for grading (do not write in this grid)

MCQ	11	12	13	14	15	16	Total
/20	/5	/5	/5	/5	/5	/5	/50

585, av. King-Edward C.P. 450, Succ. A
Ottawa (Ontario) K1N 6N5 Canada
585 King Edward Ave., P.O. Box 450, Stn. A
Ottawa, Ontario K1N 6N5 Canada
(613) 562-5864 • Téléc./Fax (613) 562-5776
Courriel/Email: uomaths@science.uottawa.ca

1. The following integral $\int_0^2 \int_0^{4-x^2} f(x,y)\,dy\,dx$ describes the integral of a function f over a type I region in the $x-y$ plane. Which of the following corresponds to the same integral, but with the region of integration viewed as a type II region instead of a type I region?

A. $\int_0^{4-x^2} \int_0^2 f(x,y)\,dx\,dy$

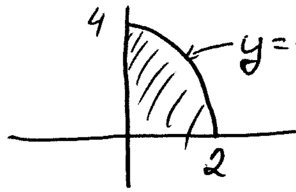
B. $\int_0^4 \int_0^{\sqrt{4-y}} f(x,y)\,dx\,dy$

C. $\int_0^4 \int_{\sqrt{4-y}}^2 f(x,y)\,dx\,dy$

D. $\int_0^{\sqrt{4-y}} \int_0^4 f(x,y)\,dy\,dx$

E. $\int_0^{\sqrt{4-y}} \int_0^4 f(x,y)\,dx\,dy$

F. $\int_0^2 \int_0^{4-y^2} f(x,y)\,dx\,dy$



$y=4-x^2 \Leftrightarrow x=\sqrt{4-y}$

So as type II

$0 \le x \le \sqrt{4-y}$

$0 \le y \le 4$

2. If $f(x,y)$ is a differentiable function such that $\vec{\nabla} f(3,4) = \vec{i} + \vec{j}$, then only one of the following lines is orthogonal to the level curve of f which passes through $(3,4)$. Which one?

A. $y = x + 1$

B. $y = 7 - x$

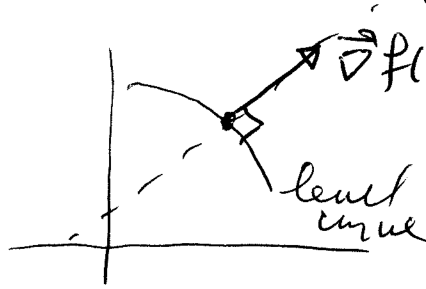
C. $y = 4$

D. $x = 3$

E. $y = 2x - 2$

F. $y = -2x + 10$

$y = x+1$



$\vec{\nabla} f(3,4) = \vec{i} + \vec{j}$ is orthogonal to level curve of f through $(3,4)$. So this vector must be along the required line. Slope must be 1.

3. Consider the surface which corresponds to the portion of the cone $z = \sqrt{x^2+y^2}$ between the planes $z=2$ and $z=3$, drawn below. What is the area of this surface?

$\vec{r}(a,\theta) = a\cos\theta\,\vec{i} + a\sin\theta\,\vec{j} + a\,\vec{k}$

$2 \le a \le 3$

$0 \le \theta \le 2\pi$

A. 5π

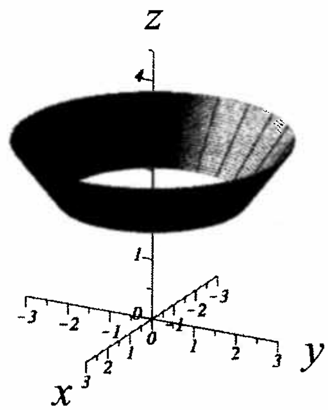
B. 2π

C. $5\sqrt{2}\,\pi$

D. $5\sqrt{2}$

E. $2\sqrt{2}\,\pi$

F. $\sqrt{2}\,\pi$



$\vec{r}_a = \cos\theta\,\vec{i} + \sin\theta\,\vec{j} + \vec{k}$

$\vec{r}_\theta = -a\sin\theta\,\vec{i} + a\cos\theta\,\vec{j}$

$$\vec{r}_a \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 1 \\ -a\sin\theta & a\cos\theta & 0 \end{vmatrix}$$

$= -a\cos\theta\,\vec{i} - a\sin\theta\,\vec{j} + a\,\vec{k}$

$\|\vec{r}_a \times \vec{r}_\theta\| = \sqrt{a^2+a^2} = \sqrt{2}\,a$

$$\text{Area} = \int_0^{2\pi}\int_2^3 \sqrt{2}\,a\,da\,d\theta = 2\pi\sqrt{2}\,\frac{a^2}{2}\Big|_2^3 = 2\pi\sqrt{2}\left(\frac{9}{2}-\frac{4}{2}\right) = 5\pi\sqrt{2}$$

4. What is the total arclength of the parametrized curve $\vec{r}(t) = (\frac{2}{3}t^{3/2})\,\vec{i} + (\cos t)\,\vec{j} + (\sin t)\,\vec{k}$, $0 \le t \le 3$?

$\vec{r}\,'(t) = t^{1/2}\,\vec{i} - \sin t\,\vec{j} + \cos t\,\vec{k}$

A. $\sqrt{t+1}$

B. 2

$\|\vec{r}\,'(t)\| = \sqrt{t + \sin^2 t + \cos^2 t} = \sqrt{t+1}$

C. $\vec{i} + \vec{k}$

D. 0

$$\text{Length} = \int_0^3 \sqrt{t+1}\,dt = \frac{2}{3}(t+1)^{3/2}\Big|_0^3$$

E. 4

F. 14/3

$= \frac{2}{3}\left(4^{3/2} - 1^{3/2}\right) = \frac{2}{3}(8-1)$

$= \frac{14}{3}$

5. If $f(x, y) = 2x^2 - 3y^2$, what is the global maximum value of f on the circle $\underbrace{x^2 + y^2 = 1}$?

$y^2 = 1 - x^2$

A. 0

B. 1

C. 2

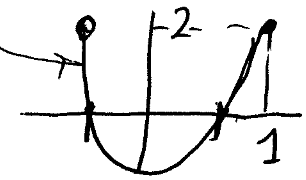
D. 3

E. 4

F. 5

$2x^2 - 3y^2 = 2x^2 - 3(1-x^2)$

$= 5x^2 - 3 \qquad -1 \le x \le 1$



6. If C is the oriented straight line segment in the plane starting at the point $(1, 1)$ and ending at the point $(2, 2)$, and $\vec{F}(x, y) = xy\,\vec{i} + 3x^2\,\vec{j}$, what is the value of the line integral $\int_C \vec{F} \cdot d\vec{r}$?

A. 9

B. 28/3

C. 29/3

D. 10

E. 31/3

F. 32/3

$\vec{r}(t) = t\vec{i} + t\vec{j} \qquad 1 \le t \le 2$

$\vec{r}\,'(t) = \vec{i} + \vec{j}$

$\vec{F}(\vec{r}(t)) = t^2\vec{i} + 3t^2\vec{j}$

$\int_C \vec{F} \cdot d\vec{r} = \int_1^2 (t^2\vec{i} + 3t^2\vec{j}) \cdot (\vec{i} + \vec{j})\,dt$

$= \int_1^2 (t^2 + 3t^2)\,dt = \int_1^2 4t^2\,dt = \frac{4t^3}{3}\Big|_1^2$

$= \frac{4 \cdot 8}{3} - \frac{4 \cdot 1}{3} = \frac{28}{3}$

7. Which of the following vector fields is conservative?

A. $\vec{F}(x,y,z) = y\,\vec{\imath} - x\,\vec{\jmath}$

B. $\vec{F}(x,y,z) = -z\,\vec{\jmath} + y\,\vec{k}$

C. $\vec{F}(x,y,z) = (x + 2y + 3z)\,\vec{\jmath}$

D. $\vec{F}(x,y,z) = 2xyz^3\,\vec{\imath} + x^2z^3\,\vec{\jmath} + 3x^2yz^2\,\vec{k}$ $\leftarrow$ is the only one for which $\vec{\nabla}\times\vec{F} = \vec{0}$

E. All of the above

F. None of the above

8. If S is the disk $x^2 + y^2 \leq 9$, $z = 4$, oriented upwards (i.e. unit normal parallel to $\vec{k}$), and $\vec{F}(x,y,z) = -x\,\vec{\imath} - y\,\vec{\jmath} - z\,\vec{k}$, what is the value of the flux integral $\iint_S \vec{F}\cdot d\vec{S}$?

A. $-36\pi\,\vec{k}$

B. 36π

C. -36π

D. 0

E. -9π

F. 9π

$$\vec{r}(a,\theta) = a\cos\theta\,\vec{\imath} + a\sin\theta\,\vec{\jmath} + 4\vec{k} \qquad 0 \leq a \leq 3,\quad 0 \leq \theta \leq 2\pi$$

$$\vec{r}_a = \cos\theta\,\vec{\imath} + \sin\theta\,\vec{\jmath} \qquad \vec{r}_\theta = -a\sin\theta\,\vec{\imath} + a\cos\theta\,\vec{\jmath}$$

$$\vec{r}_a \times \vec{r}_\theta = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \cos\theta & \sin\theta & 0 \\ -a\sin\theta & a\cos\theta & 0 \end{vmatrix} = a\vec{k}$$

$$\vec{F}(\vec{r}(a,\theta)) = -a\cos\theta\,\vec{\imath} - a\sin\theta\,\vec{\jmath} - 4\vec{k}$$

$$\iint_S \vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^3 (-a\cos\theta\,\vec{\imath} - a\sin\theta\,\vec{\jmath} - 4\vec{k})\cdot a\vec{k}\,da\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 -4a\,da\,d\theta = -8\pi \frac{a^2}{2}\Big|_0^3 = -8\pi\cdot\frac{9}{2} = -36\pi$$

9. Consider the two-dimensional region D drawn below, whose boundary is the oriented curve C, also drawn. Let $\vec{F}(x,y) = P(x,y)\,\vec{\imath} + Q(x,y)\,\vec{\jmath}$ be a vector field with continuous partial derivatives. Then which of the following equations corresponds to Green's theorem?

A. $\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x}\right) dx\,dy$

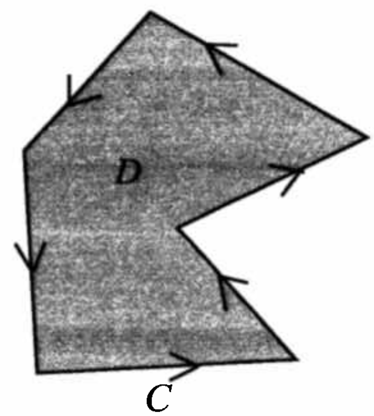
B. $\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}\right) dx\,dy$

C. $\int_C Q\,dx + P\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

D. $\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dx\,dy$

E. $\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y}\right) dx\,dy$

F. $\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$



See notes

10. If $f(x,y) = xe^y$ and $\vec{u} = \dfrac{\vec{\imath} + 2\vec{\jmath}}{\sqrt{5}}$, then the directional derivative $D_{\vec{u}} f(2,0)$ is equal to

A. 1

B. $\sqrt{2}$

C. $\sqrt{3}$

D. 2

E. $\sqrt{5}$

F. $\sqrt{6}$

$f_x = e^y \qquad f_y = xe^y$

$\vec{\nabla} f(2,0) = \vec{\imath} + 2\vec{\jmath}$

$D_{\vec{u}} f(2,0) = \vec{\nabla} f(2,0) \cdot \vec{u} = (\vec{\imath} + 2\vec{\jmath}) \cdot \left(\frac{\vec{\imath} + 2\vec{\jmath}}{\sqrt{5}}\right)$

$= \frac{5}{\sqrt{5}} = \sqrt{5}$

$$\overbrace{}^{P} \qquad \overbrace{}^{Q}$$

11. Consider the vector field $\vec{F}(x,y) = \underbrace{(e^x y^2 + 1)}_{P}\,\vec{\imath} + \underbrace{2e^x y}_{Q}\,\vec{\jmath}$. Show that the vector field is conservative, then find a scalar function $f(x,y)$ such that $\vec{F}(x,y) = \vec{\nabla} f(x,y)$. Finally, compute $\int_C \vec{F} \cdot d\vec{r}$, where C is some smooth oriented curve which starts at the point $(1,2)$ and finishes at the point $(2,1)$.

$$\frac{\partial Q}{\partial x} = 2e^x y = \frac{\partial P}{\partial y} = 2e^x y$$

$\vec{F}$ is Conservative.

So there exists a potential f s.t. $\vec{F} = \vec{\nabla} f$.

$$\frac{\partial f}{\partial x} = e^x y^2 + 1 \Rightarrow f(x,y) = e^x y^2 + x + h(y)$$

$$\frac{\partial f}{\partial y} = 2e^x y + h'(y) = Q = 2e^x y$$

$$\Rightarrow h'(y) = 0 \Rightarrow h(y) = K$$

So a potential is $f(x,y) = e^x y^2 + x$

Since $\vec{F}$ is Conservative, then

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{\nabla} f \cdot d\vec{r} = f(2,1) - f(1,2)$$

$$= (e^2 \cdot 1^2 + 2) - (e^1 \cdot 2^2 + 1)$$

$$= e^2 - 4e + 1$$

12. Consider the vector field $\vec{F}(x,y,z) = (z^2 - x)\,\vec{i} + (x^2 - y)\,\vec{j} + (y^2 - z)\,\vec{k}$.

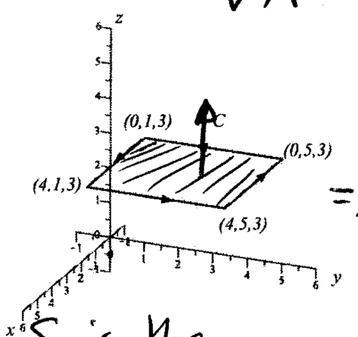
(a) Compute the divergence and the curl of $\vec{F}$, i.e. compute $\vec{\nabla} \cdot \vec{F}$ and $\vec{\nabla} \times \vec{F}$.

(b) **Using Stokes' theorem**, compute the line integral $\int_C \vec{F} \cdot d\vec{r}$, where C is the square path that starts at the point $(0,1,3)$, goes to $(4,1,3)$, then onto $(4,5,3)$, then to $(0,5,3)$, and finally back again to the starting point at $(0,1,3)$.

$$\vec{\nabla} \cdot \vec{F} = -1 - 1 - 1 = -3$$

0.5

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z^2 - x & x^2 - y & y^2 - z \end{vmatrix}$$



$$= \vec{i}\,(2y) - \vec{j}\,(-2z) + \vec{k}\,(2x)$$

$$= 2y\,\vec{i} + 2z\,\vec{j} + 2x\,\vec{k}$$

By Stokes' thm, we have

$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}$ where S is the square $0 \le x \le 4$, $1 \le y \le 5$, $z = 3$, oriented upwards

$\vec{r}(x,y) = x\,\vec{i} + y\,\vec{j} + 3\,\vec{k}$, $0 \le x \le 4$, $1 \le y \le 5$ $\quad \vec{r}_x \times \vec{r}_y = \vec{i} \times \vec{j} = \vec{k}$

$$(\vec{\nabla} \times \vec{F})(\vec{r}(x,y)) = 2y\,\vec{i} + 6\,\vec{j} + 2x\,\vec{k}$$

$$\iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = \int_1^5 \int_0^4 (2y\,\vec{i} + 6\,\vec{j} + 2x\,\vec{k}) \cdot \vec{k}\, dx\, dy = \int_1^5 \int_0^4 2x\, dx\, dy$$

$$= \int_1^5 x^2 \Big|_0^4 dy = \int_1^5 16\, dy = 16 \cdot 4 = 64$$

(Extra page)

13. Consider the vector field $\vec{F}(x,y,z) = (xz + \sin y)\,\vec{\imath} + (y^3 + z)\,\vec{\jmath} + (3e^x - 3zy^2)\,\vec{k}$.

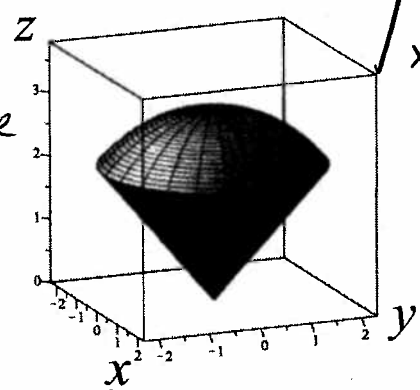
(a) Compute the divergence and the curl of $\vec{F}$, i.e. compute $\vec{\nabla}\cdot\vec{F}$ and $\vec{\nabla}\times\vec{F}$.

(b) Let E denote the three-dimensional solid region below the hemisphere $z = \sqrt{9 - x^2 - y^2}$ and above the cone $z = \sqrt{x^2 + y^2}$ (illustrated below), and let S denote the surface which is the boundary of E (i.e. S is the "skin" of E), oriented outwards. Compute the flux integral $\iint_S \vec{F}\cdot d\vec{S}$ using any method you prefer. Note that although there are a few different ways to compute this, some methods will yield much simpler computations than other methods, so choose wisely. In particular, some of the computations you did in (a) above may be useful.

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz+\sin y & y^3+z & 3e^x-3zy^2 \end{vmatrix}$$

$$\vec{\nabla}\cdot\vec{F} = z + 3y^2 - 3y^2 = z$$

$$= \vec{\imath}(-6zy-1) - \vec{\jmath}(3e^x - x) + \vec{k}(-\cos y)$$



This is a situation where Gauss' divergence thm applies. So

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \vec{\nabla}\cdot\vec{F}\, dV$$

where E is the solid which is described in spherical coordinates as

$0 \le \rho \le 3$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi/4$

So $\iint_S \vec{F}\cdot d\vec{S} = \iiint_E z\, dV$

$$= \int_0^{\pi/4}\int_0^{2\pi}\int_0^3 \underbrace{\rho\cos\phi}_{z}\ \underbrace{\rho^2\sin\phi\, d\rho\, d\theta\, d\phi}_{dV}$$

$$= \cdots = \frac{81\pi}{8}$$

(Extra page)

14. Find and classify the critical points of the function $f(x, y) = xy(20 - 4x - 5y)$.

$f_x = y(20 - 8x - 5y)$ $\quad f_y = 2x(10 - 2x - 5y)$

The system $f_x = 0$, $f_y = 0$ has 4 solutions :

$(0,0)$, $(0,4)$, $(5,0)$, $(5/3, 4/3)$.

$f_{xx} = -8y$ $\quad f_{xy} = f_{yx} = 20 - 8x - 10y$ $\quad f_{yy} = -10x$

$D = f_{xx} f_{yy} - (f_{xy})^2 = 80xy - (20 - 8x - 10y)^2$

$(0,0)$: $D = -20^2 = -400 < 0 \Rightarrow (0,0)$ is a saddle

$(5,0)$: $D = -(20-40)^2 = -400 < 0 \Rightarrow (5,0)$ is a saddle

$(0,4)$: $D = -(20-40)^2 = -400 < 0 \Rightarrow (0,4)$ is a saddle

$(5/3, \frac{4}{3})$: $D = \frac{400}{3} > 0$, $f_{xx} = -8 \cdot \frac{4}{3} < 0$

$\Rightarrow (\frac{5}{3}, \frac{4}{3})$ is is local max.

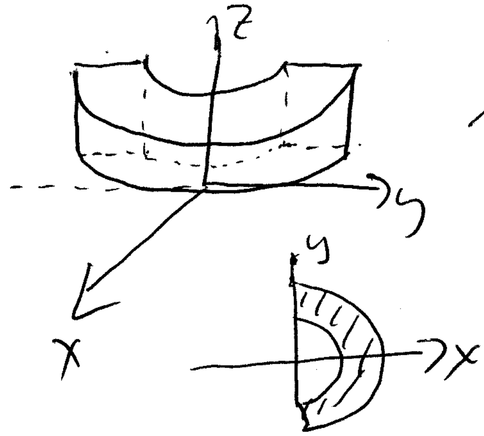
15. Consider the solid object that is defined by the inequalities

$$x \geq 0$$

$$1 \leq x^2 + y^2 \leq 4$$

$$1 \leq z \leq 3.$$

This solid has a mass density given by $\delta(x, y, z) = xz$. Set up a triple integral in cylindrical coordinates that gives the total mass of this solid, but **do not evaluate this integral**.



In cylindrical coordinates, this region is described as

$$1 \leq r \leq 2$$

$$1 \leq z \leq 3$$

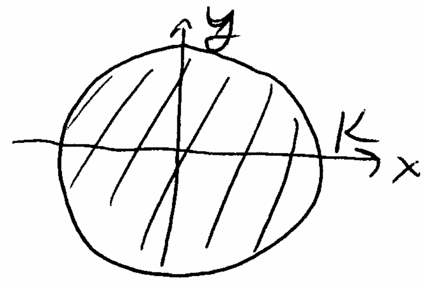
$$-\pi/2 \leq \theta \leq \pi/2$$

So $\text{Mass} = \iiint_E \delta \, dV = \int_{-\pi/2}^{\pi/2} \int_1^3 \int_1^2 r\cos\theta \, z \, r \, dr \, dz \, d\theta$

$$= \boxed{\int_{-\pi/2}^{\pi/2} \int_1^3 \int_1^2 r^2 z \cos\theta \, dr \, dz \, d\theta}$$

16. Let $K > 0$ be some positive number, and consider the disk of radius K centered at the origin: $D = \{\,(x,y)\,|\,x^2 + y^2 \leq K^2\,\}$. Compute

$$\iint_D e^{-(x^2+y^2)}\,dA$$



In polar coordinate D is described as

$0 \leq r \leq K$

$0 \leq \theta \leq 2\pi$

$$\iint_D e^{-(x^2+y^2)}\,dA = \int_0^{2\pi}\int_0^{K} e^{-r^2}\, r\,dr\,d\theta$$

$$= \int_0^{2\pi} -\frac{1}{2}e^{-r^2}\Big|_0^K\,d\theta = \int_0^{2\pi}\left(-\frac{1}{2}e^{-K^2} + \frac{1}{2}\right)d\theta$$

$$= 2\pi\left(\frac{1}{2} - \frac{1}{2}e^{-K^2}\right) = \pi\left(1 - e^{-K^2}\right)$$

(Extra page)